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Invariant Measures for a Two-Species Asymmetric Process

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We consider a process of two classes of particles jumping on a one-dimensional lattice. The marginal system of the first class of particles is the one-dimensional totally asymmetric simple exclusion process. When classes are disregarded the process is also the totally asymmetric simple exclusion process. The existence of a unique invariant measure with product marginals with density ρ and λ for the first- and first- plus second-class particles, respectively, was shown by Ferrari, Kipnis, and Saada. Recently Derrida, Janowsky, Lebowitz, and Speer have computed this invariant measure for finite boxes and performed the infinite-volume limit. Based on this computation we give a complete description of the measure and derive some of its properties. In particular we show that the invariant measure for the simple exclusion process as seen from a second-class particle with asymptotic densities ρ and λ is equivalent to the product measure with densities ρ to the left of the origin and λ to the right of the origin.

KEY WORDS: Two-species process; asymmetric simple exclusion; second-class particles.

1. INTRODUCTION

The simplest way of defining the two-species system is by using the basic coupling of the totally asymmetric simple exclusion process (SEP). We define the *simple exclusion process* $\eta_t \in \{0, 1\}^{\mathbb{Z}}$ $(t \geqslant 0)$ as follows. At each site $x \in \mathbb{Z}$ we attach a random clock that rings according to a Poisson process of parameter 1. The clocks are mutually independent. When the clock of an occupied site x rings, if $x+1$ is empty, the particle at x jumps to $x+1$. If $x+1$ is occupied, nothing happens. Thus, in this process, the particles are generally drifting to the right. If one considers two initial configurations η^1 and $\eta^2 \in \{0, 1\}^{\mathbb{Z}}$ such that $\eta^1(x) \leqslant \eta^2(x)$ for all x, and uses

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the same clocks for both realizations, then one has a coupled process (η_t^1, η_t^2) with the property that $\eta_t^1(x) \leqslant \eta_t^2(x)$ for all x and all $t \geqslant 0$.[12,13] The two-species process (σ_t, ξ_t) $(t \geqslant 0)$ is defined by putting $\sigma_t(x) = \eta_t^1(x)$ and $\xi_t(x) = \eta_t^2(x) - \eta_t^1(x)$. The σ particles are the *first-class* particles and the ξ particles are the *second-class* particles. The reason for these terms is that, when a clock rings for a first-class particle at site x and a second-class particle is at site $x+1$, the particles interchange positions, whereas if a second-class particle is at x and a first-class particle is at $x+1$, they do *not* move. It is easy to see that the two-species process is Markovian.

In this paper we are concerned with the invariant measures for the two-species process. Since the marginal processes σ_t and $\sigma_t + \xi_t$ are simple exclusion processes, the corresponding marginal measures of any invariant measure for the two-species process must be invariant for the SEP. Now, the invariant measures for the SEP are convex combinations of the product measures ν_ρ with density $\rho \in [0, 1]$ and the blocking measures concentrated on the configuration ...000111... and its translates. Let us say that a distribution of (σ, ξ) has *good marginals* if, for some $\rho \leqslant \lambda$, its σ marginal is ν_ρ and its $\sigma + \xi$ marginal is ν_λ. It is easy to construct a product measure π_2 for (σ, ξ) with good marginals. Ferrari *et al.*[9] proved that, for the two-species process (σ_t, ξ_t), there exists a unique invariant measure μ_2 with good marginals, and that the process started with the product measure π_2 converges to μ_2 as $t \to \infty$. Derrida *et al.*[4] have recently computed the invariant measure μ_2 in finite boxes and, performing the infinite-volume limit, they have investigated μ_2. In a sequel, Speer[17] makes this approach rigorous.

One important fact discovered by Derrida *et al.*[4] is that, under the invariant measure μ_2, the distribution to the right of a second-class particle is *independent* of the distribution to its left. This suggests studying the process "as seen from a second-class particle." To make this precise, assume that at time $t = 0$ there is a second-class particle at the origin and let X_t be its position at time t. The process *as seen from this second-class particle* is $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$, where τ_x denotes translation by x. Thus,

$$(\tau_{X_t}\sigma_t(x), \tau_{X_t}\xi_t(x)) = (\sigma_t(x + X_t), \xi_t(x + X_t)) \quad \text{for all } x \in \mathbb{Z}$$

Now assume that the initial distribution μ of the two-species process is translation invariant, and that it has a positive density of second-class particles. Then, at time t, the process as seen from a second-class particle, started with the measure μ conditioned to having a second-class particle at the origin, has the same distribution as the two-species process started with the unconditioned measure μ, but itself conditioned to having a second-class particle at the origin at time t. This means that, when the density of second-class particles is positive, the invariant measures for the two-species process have a corresponding invariant measure for the process as seen from a second-class particle.[9] But in fact the process as seen from a second-class particle is richer: it has invariant measures with only a finite number of second-class particles with no corresponding measure in the two-species process.

Our main contribution is a complete description of the invariant measures for the two-species process as seen from a fixed second-class particle. This description is based on computations in Derrida *et al.*[4] We consider two densities $0<\rho\leqslant\lambda<1$ and construct a measure $\mu'_2=\mu'_2(\rho,\lambda)$ that is invariant for our process (see Theorem 1). The parameters correspond to the asymptotic densities (as $x\to\pm\infty$) of the first- and the first-plus second-class particles. The cases $\rho=0$ or $\lambda=1$ are easier and were considered before. In the particular case in which $\rho=0$ or $\lambda=1$, either the σ marginal or the $\sigma+\xi$ marginal is trivial. Moreover, in this case, the other marginal is trivial if $\rho=\lambda$ and it is the SEP if $\rho<\lambda$. In the latter case, the process corresponds to the SEP as seen from a tagged particle, as studied by Ferrari[6] and De Masi *et al.*[3] An important general property of the measure μ'_2 is "translation invariance," in the sense that it is the same seen from *any* second-class particle. When $\rho<\lambda$ this "translation invariance" implies that there exists a unique translation-invariant μ_2 such that μ'_2 is μ_2 conditioned to having a second-class particle at the origin. When $\lambda=\rho$, the average distance between two successive second-class particles is infinite. This implies that there is no translation-invariant measure μ_2 such that μ'_2 is μ_2 conditioned to having a second-class particle at the origin.

Recall that our particles are drifting to $+\infty$. The definition of the two-species process is such that a first-class particle can overtake a second-class one, but the other way around is prohibited. Let us start the process with a second-class particle at the origin, and, along the evolution of the process, we refer to this particle as the *zeroth second-class particle.* We consider the second-class particles from left to right, so that we may speak of the ith second-class particle for any $i\in\mathbb{Z}$. If one identifies the two classes of particles starting from (and to the right of) the ith second-class particle ($i>0$), one has an operator Φ_i acting on the configurations (σ,ξ), which commutes with the semigroup corresponding to the evolution. Similarly, for any fixed $j<0$, one can identify *holes*, i.e., empty sites, with second-class particles starting from, and to the left of, the jth second-class particle to obtain an operator Ψ_j that also commutes with the semigroup. Hence, applying any (or both) of these operators to the "translation-invariant" stationary measure μ'_2, we obtain another invariant measure (see Theorem 2). Incidentally, these new measures are clearly not "translation invariant." Now, identifying first- and second-class particles to the right of the particle at the origin and holes and second-class particles to the left of it, we obtain the invariant measure for the process as seen from a single, isolated second-class particle. When $\rho < \lambda$ this corresponds to a shock in the SEP.[7,9] When $\rho = \lambda$ there is a reminiscence of the shock, as the density to the right of the second-class particle is bigger than the density to the left of it and the approach to the asymptotic density λ, which equals ρ here, is slow.[4] If one makes the identification for all but two second-class particles, one gets that the distance d between the two second-class particles is, following the terminology of Derrida *et al.*,[4] a "bounded state" even when $\lambda = \rho$. Indeed, the distribution of d is the same as the distribution of the distance between two successive second-class particles under μ'_2. It turns out that this distance has the same distribution as the hitting time of 1 for a nearest-neighbor random walk with jumps in $\{-1, 0, 1\}$ with probabilities $\rho(1-\lambda)$, $1-\lambda(1-\rho)-\rho(1-\lambda)$, and $\lambda(1-\rho)$, respectively (see Lemma 2.5).

Our approach relies on the work of Derrida *et al.*[4] and Speer,[17] but we work directly in the infinite volume. In Section 3 we describe completely the measure μ'_2 and show that it is invariant for the process as seen from a fixed second-class particle. Derrida *et al.*[4] state the following remarkable property of the measure μ'_2: the distribution of first-class particles to the right of the tagged second-class particle is the product measure ν_ρ with density ρ, while the distribution of empty sites to the left of the tagged second-class particle is the product measure $\nu_{1-\lambda}$ with density $1-\lambda$. Speer[17] proves this statement and here we give an alternative proof of this fact by showing that one may construct μ'_2 as follows. We first put a second-class particle at the origin and distribute the first-class particles to the right of the origin according to the measure ν_ρ. Then we give a recipe for deciding where to put the second-class particles among the unoccupied sites. To the left of the second-class particle at the origin the positions of the empty sites are chosen according to the product measure $\nu_{1-\lambda}$ and a similar recipe is used to decide where to put the second-class particles. (See Proposition 1.)

When $\lambda > \rho$ there exists a unique translation-invariant measure μ_2 with the property that it coincides with μ'_2 when it is conditioned to having a second-class particle at the origin. As explained above, the invariance of μ'_2 for the process as seen from the second-class particle implies that the measure μ_2 is invariant for the two-species process. Using the property that the first-class particles to the right of the tagged second-class particle are distributed according to a product measure, we show that μ_2 has good marginals (cf. Theorem 3). This already followed from the infinite-volume limit of Derrida,[4] but in a somewhat indirect way. We also show that it is possible to construct a coupling $\bar{\mu}$ with marginals μ'_2 and μ_2 in such a way that the number of sites where the two marginals differ is a random variable with a finite exponential moment. (See Theorem 4.)

Let $\nu_{\rho,\lambda}$ be the product measure with density ρ to the left of the origin and density λ to the right of the origin. Using the results of Ferrari *et al.*,[19] Ferrari[7] proved that the SEP as seen from a second-class particle starting with the product measure $\nu_{\rho,\lambda}$ presents a shock: uniformly in time the asymptotic densities are ρ and λ to the left and right of the origin, respectively. Indeed the process with a unique second-class particle, started at the origin, with initial product distribution $\nu_{\rho,\lambda}$ can be coupled to the two-species process with initial product distribution π_2 (with marginals ν_ρ and ν_λ) in such a way that at all times the single second-class particle of the first process has the same position as the tagged second-class particle in the second process. As mentioned above, this can be done by identifying first- and second-class particles to the right of the origin and empty sites and second-class particles to the left of the origin. Applying the results for the two-species process to the shock in the SEP, Derrida *et al.*[4] have computed the rate of convergence of the density of the shock to the asymptotic densities ρ and λ. We make a further step proving that the invariant measure μ' for the process as seen from a single second-class particle has the following property. One may construct a coupling between μ' and $\nu_{\rho,\lambda}$ in such a way that the number of sites where the two marginals differ is a random variable with a finite exponential moment. This implies in particular that μ' is equivalent to $\nu_{\rho,\lambda}$. (See Theorem 5 and its corollary.)

Let us now mention some related results. Speer[17] described the set of all invariant measures for the two-species process and showed that the invariant measure μ_2 is not Gibbsian. Ferrari and Fontes[8] computed the asymptotic variance of the position of the second-class particle for the process with initial distribution μ_2', and they studied the density fluctuation fields for the exclusion process with a shock initial condition.

This article is organized as follows. In the next section we prove three basic lemmas (Lemmas 2.1–2.3) that are used in later sections. In Section 3 we give our construction of the measure μ_2' and we prove Theorem 1, which asserts that μ_2' is invariant for the process $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$. Also in this section are Theorem 2, concerning other invariant measures constructed from μ_2' with the aid of the operators Φ_i and Ψ_j, and Proposition 1. In Section 4 we deal with the invariant measure μ_2 for the process (σ_t, ξ_t), and prove that it has good marginals (cf. Theorem 3). In that section we also prove Theorem 4, concerning the coupling $\tilde{\mu}$ between μ_2 and μ_2' mentioned above. The last section is devoted to proving Theorem 5, on the coupling between μ' and $\nu_{\rho,\lambda}$.

2. A DISTRIBUTION ON THE SET OF FINITE CONFIGURATIONS

Let $\mathbf{Y}$ be the space of *finite configurations* of 0's and 1's, i.e.,

$$\mathbf{Y} = \bigcup_{n \geqslant 0} \{0, 1\}^n = \{\varnothing, 0, 1, 00, 01, 10, 11, 000, \dots\}$$

Usually, we think of a sequence ζ in $\mathbf{Y}$ of length n as indexed by $\{1, \dots, n\}$. In this section we define and study a certain probability distribution p on the space $\mathbf{Y}$. This distribution will be used in the next section to construct the invariant measure for the system as seen from a second-class particle.

Let $\zeta \in \mathbf{Y}$ be given. We write $N(\zeta)$ for the length of ζ, and $K(\zeta)$ for the number of 1's in ζ. Formally, we have $N(\zeta) = n$ if and only if $\zeta \in \{0, 1\}^n$, and $K(\zeta) = \sum_{x=1}^{N(\zeta)} \zeta(x)$. An important definition that we shall need is the following. For $\zeta \in \mathbf{Y}$, let $M(\zeta)$ be the number of *distinct configurations that can be obtained from ζ by shifting ones to the right, including ζ itself.* Thus, for example, we have $M(100) = 3$, $M(0011) = 1$, and $M(1010) = 5$.

We may now define the distribution p on $\mathbf{Y}$. In fact, we shall define a distribution $p = p_{\rho,\lambda}$ for each $0 < \rho \leqslant \lambda < 1$. Let ρ and λ as above be fixed. Given $\zeta \in \mathbf{Y}$, we put

$$p(\zeta) = p_{\rho,\lambda}(\zeta) = \lambda(1-\rho)\, M(\zeta)(\lambda\rho)^{K(\zeta)}\, [(1-\lambda)(1-\rho)]^{N(\zeta) - K(\zeta)} \tag{2.1}$$

We show in Lemma 2.1(i) below that p does indeed define a probability distribution over $\mathbf{Y}$. It is with the aid of $p = p_{\rho,\lambda}$ that we shall construct the invariant measure for the two-species process as seen from a second-class particle when the asymptotic densities of the first-class particles and the first- plus second-class particles are, respectively, ρ and λ.

The rest of this section is devoted to proving that p gives a probability measure over $\mathbf{Y}$ and to the study of some simple properties of the space $(\mathbf{Y}, p)$ and of the function $M(\zeta)$ ($\zeta \in \mathbf{Y}$). In particular, we shall consider the random variable $N = N(\zeta)$, that is, the random length of a sequence ζ drawn from $\mathbf{Y}$ according to p. The main results of this section are given in Lemmas 2.1–2.3, which we now state.

Lemma 2.1. Let $\rho, \lambda \in (0, 1)$ be fixed. Then (i) $\sum_{\zeta \in \mathbf{Y}} p(\zeta) = 1$ if and only if $\rho \leqslant \lambda$. Assuming that $\rho \leqslant \lambda$ and, writing for the expectation in $(\mathbf{Y}, p)$, we have (ii) if $\rho < \lambda$, then $\mathbb{E}(N+1) = 1/(\lambda - \rho)$, and (iii) if $\rho = \lambda$, then $\mathbb{E}(N+1) = \infty$. Finally, (iv) if $\rho < \lambda$, then N has a finite exponential moment. In other words, there exists $\theta > 0$ such that

$$\mathbb{E}^{\theta N} < \infty \tag{2.2}$$

The distribution of the random variable N is given in Lemma 2.5 below. The generating function of N is given in the following lemma.

Lemma 2.2. Let $\rho \leqslant \lambda$. The generating function of N is given by

$$\mathbb{E}s^N = \frac{1}{2as^2}\{1 - cs - [(1-cs)^2 - 4abs^2]^{1/2}\} \tag{2.3}$$

where $a = \rho(1-\lambda)$, $b = \lambda(1-\rho)$, and $c = 1 - a - b$. The closed disc $|s| \leqslant \{1 - (\sqrt{b} - \sqrt{a})^2\}^{-1}$ is its domain of convergence.

Our next lemma, Lemma 2.3, is inspired by Derrida *et al.*[4]

Lemma 2.3. For all $\zeta, \gamma \in \mathbf{Y}$, we have $M(\zeta 10\gamma) = M(\zeta 1\gamma) + M(\zeta 0\gamma)$.

We now define a random walk on the integers that will be important in the sequel. Let $X_1, X_2,..., Y_1, Y_2,...,$ be independent 0–1 random variables with $\mathbb{E}(X_i) = \lambda$ and $\mathbb{E}(Y_i) = \rho$ $(i \geqslant 1)$. Put $Z_i = X_i - Y_i$ $(i \geqslant 1)$, and let $\tilde{Z}_n = \sum_{1 \leqslant i \leqslant n} Z_i$ $(n \geqslant 0)$. Note that then $(\tilde{Z}_n)_0^\infty$ is a random walk on $\mathbb{Z}$, and let $T = \inf\{n \geqslant 0: \tilde{Z}_n = 1\}$ be the hitting time of the event $\{\tilde{Z}_n = 1\}$.

The rest of this section is devoted to proving the lemmas above. The other sections of this paper may be read independently from what follows. Our first auxiliary lemma is the following.

Lemma 2.4. For all integers $n, k \geqslant 0$, we have

$$\sum M(\zeta) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} \tag{2.4}$$

where the sum ranges over all $\zeta \in \mathbf{Y}$ with $N(\zeta) = n$ and $K(\zeta) = k$.

We defer the proof of Lemma 2.4 until later, and pass on to a result that is crucial in the proof of Lemma 2.1(i).

For a finite configuration $\zeta \in \mathbf{Y}$, recall that $N(\zeta)$ denotes its length and $K(\zeta)$ its number of 1's. For integers n and k, set $p_{n,k} = \sum_\zeta p(\zeta)$, where the sum ranges over all $\zeta \in \mathbf{Y}$ with $N(\zeta) = n$ and $K(\zeta) = k$. Note that once we know that p is a probability measure on $\mathbf{Y}$, the quantity $p_{n,k}$ is simply the probability that a random configuration $\zeta \in \mathbf{Y}$ has length n and k 1's. In particular, the lemma below in this case simply states that $\mathbb{P}\{N = n\} = \mathbb{P}\{T = n+1\}$.

Lemma 2.5. Let $\lambda, \rho \in (0, 1)$ be fixed. Then, for any $n \geqslant 0$, we have

$$\sum_k p_{n,k} = \mathbb{P}\{T = n+1\} \tag{2.5}$$

Proof. By Lemma 2.4, we have

$$p_{n,k}=\frac{1}{n+1}\binom{n+1}{k+1}\lambda^{k+1}(1-\lambda)^{n-k}\binom{n+1}{k}\rho^{k}(1-\rho)^{n+1-k}$$
$$=\frac{1}{n+1}\mathbb{P}(W_1=k+1)\,\mathbb{P}(W_2=k)$$

where W_1 and W_2 are two independent binomial random variables with parameters $n+1$ and λ and $n+1$ and ρ, respectively. Now, summing over all k, for $n\geqslant 0$ we have

$$\sum_k p_{n,k}=\frac{1}{n+1}\mathbb{P}\{W_1-W_2=1\}=\frac{1}{n+1}\mathbb{P}\{\tilde{Z}_{n+1}=1\} \tag{2.6}$$

where $(\tilde{Z}_n)_0^\infty$ is the random walk introduced above. On the other hand, recalling that T is the hitting time of 1 for that walk, we have

$$\mathbb{P}\{T=n+1\}=\frac{1}{n+1}\mathbb{P}\{\tilde{Z}_{n+1}=1\} \tag{2.7}$$

for all integers $n\geqslant 0$. Identity (2.7) is Exercise (IV.12) of Spitzer,[18] but for completeness we give a combinatorial proof for it in Lemma 2.6 below. Lemma 2.5 follows from (2.6) and (2.7). ■

Remark 2.1. The Local Central Limit Theorem (or direct calculations) and (2.7) imply that, when $0<\rho=\lambda<1$, we have $\mathbb{P}(T=n)=[c+o(1)]\,n^{-3/2}$ as $n\to\infty$, where $c=c(\rho)>0$ depends only on ρ. For the case in which $\rho<\lambda$, see Remark 2.2.

Let us now prove (2.7). The proof below is entirely combinatorial and more elementary than the one suggested in Spitzer,[18] which is based on Lagrange's inversion formula.

Lemma 2.6. Let $(V_i)_1^\infty$ be a family of i.i.d. $\{\pm 1, 0\}$-random variables and let $\tilde{V}_n=\sum_{1\leqslant i\leqslant n}V_i\,(n\geqslant 0)$ be the associated random walk on $\mathbb{Z}$. Let $T=\inf\{n\colon \tilde{V}_n=1\}$ be the hitting time of $\{\tilde{V}_n=1\}$. Then $\mathbb{P}\{T=n\}=n^{-1}\,\mathbb{P}\{W_n=1\}$ for all integers $n\geqslant 1$.

Proof. We deduce this result from a lemma of Raney[16] (see also Example 4 in Section 7.5 of ref. 10): if $\mathbf{x}=(x_1,\dots,x_n)$ is a sequence of integers with $\sum_{1\leqslant i\leqslant n}x_i=1$, then there is a unique cyclic permutation of $\mathbf{x}$, say $(x_j, x_{j+1},\dots, x_n, x_1,\dots, x_{j-1})$, all of whose proper initial partial sums are nonpositive, i.e., such that $x_j,\ x_j+x_{j+1},\dots, x_j+\cdots+x_{j-2}\leqslant 0$.

Let $\mathbf{x} = (x_1, ..., x_n)$ be a $\{\pm 1, 0\}$-sequence with $\sum_{1 \leqslant i \leqslant n} x_i = 1$, and let $E_\mathbf{x}$ be the event that $(V_1, ..., V_n)$ is a cyclic permutation of $\mathbf{x}$. It is simple to check, and in fact it follows from Raney's lemma, that all the n cyclic permutations of $\mathbf{x}$ are distinct. Also, clearly, the probability that $(V_i)_1^n$ is any of these n permutations is $(1/n)\, \mathbb{P}(E_\mathbf{x})$. Now, by Raney's lemma, exactly one of these permutations corresponds to the event $\{T = n\}$, and hence Lemma 2.6 follows. ■

We may now prove Lemma 2.1, the first main result of this section.

Proof of Lemma 2.1. (i) We need to prove that $\sum_{n,k} p_{n,k} = 1$ if and only if $\rho \leqslant \lambda$. In view of (2.5), we have $\sum_{n,k} p_{n,k} = \sum_n \mathbb{P}(T = n+1) = \mathbb{P}(T < \infty)$, where T is the hitting time of 1 for the walk $(\tilde{Z}_n)_1^\infty$ defined just after Lemma 2.3. It now suffices to notice that $T < \infty$ almost surely if and only if the walk $(\tilde{Z}_n)_0^\infty$ has nonnegative drift. This proves (i).

We assume from now on that $\rho \leqslant \lambda$, and rewrite (2.5) as $\mathbb{P}\{N = n\} = \mathbb{P}\{T = n+1\}$ $(n \in \mathbb{Z})$. Let us now prove (ii). Suppose that $\rho < \lambda$. Then, again considering the random walk $\tilde{Z}_n = \sum_{1 \leqslant i \leqslant n} Z_i$ and the hitting time T, by Wald's identity we obtain $1 = \mathbb{E}(\tilde{Z}_T) = \mathbb{E}(Z_i)\, \mathbb{E}(T) = (\lambda - \rho)\, \mathbb{E}(T)$, and hence $\mathbb{E}(N+1) = \mathbb{E}(T) = 1/(\lambda - \rho)$, as required.

To see (iii), note that for $\lambda = \rho$ the expected hitting time $\mathbb{E}T$ is infinite. Finally, to prove (iv), we prove that $\mathbb{P}\{N = n\}$ decays exponentially with n. By (2.5) and (2.7), we have

$$\begin{aligned} \mathbb{P}\{N = n\} &= \mathbb{P}\{T = n+1\} = (n+1)^{-1}\, \mathbb{P}\{\tilde{Z}_{n+1} = 1\} \\ &\leqslant \mathbb{P}\{\tilde{Z}_{n+1} = 1\} \end{aligned}$$

for all integers $n \geqslant 0$. But then it suffices to notice that this last probability is exponentially small, since $\mathbb{E}(Z_i) = \lambda - \rho > 0$. Indeed, if n is large enough with respect to $\lambda - \rho$, we have that

$$\mathbb{P}(\tilde{Z}_{n+1} = 1) \leqslant \exp\left\{ -\frac{(\lambda - \rho)^2}{5(\lambda - \rho + 1)}\, n \right\} \tag{2.8}$$

by Hoeffding's inequality.[11,15] ■

Proof of Lemma 2.2. The result is obtained by a standard application of Wald's identity to the stopping time T,[2] and standard analytic continuation arguments. ■

Remark 2.2. It follows from Lemma 2.2 that $\mathbb{P}\{N = n\}$ decays a little faster than is suggested in (2.8) in the proof of Lemma 2.1(iv). The rate of exponential decay of the distribution of N when $\lambda > \rho$ is given by

$$\limsup_{n \to \infty} \mathbb{P}\{N = n\}^{1/n} = 1 - \{[\lambda(1-\rho)]^{1/2} - [\rho(1-\lambda)]^{1/2}\}^2$$

We now turn to the proof of Lemma 2.3. Let $\zeta \in \mathbf{Y}$ be given. Write $\mathcal{M}(\zeta)$ for the set of configurations that can be obtained from ζ by translating ones to the right. Thus $M(\zeta)$ is simply the cardinality $|\mathcal{M}(\zeta)|$ of $\mathcal{M}(\zeta)$. If $\eta \in \mathbf{Y}$, then $\eta\zeta$ will denote the sequence in $\mathbf{Y}$ obtained by the concatenation of η and ζ. Finally, if $\mathbf{X} \subset \mathbf{Y}$, we let $\mathbf{X}\zeta = \{\eta\zeta : \eta \in \mathbf{X}\}$.

Proof of Lemma 2.3. We fix $\zeta \in \mathbf{Y}$, and use induction on $N(\gamma)$. If $N(\gamma) = 0$, that is, if γ is the empty sequence, then it suffices to notice that $\mathcal{M}(\zeta 10) = \mathcal{M}(\zeta 1)\, 0 \cup \mathcal{M}(\zeta 0)\, 1$, where the union is clearly disjoint. Thus the result follows in this case. Assume now that $N(\gamma) \geqslant 1$, and that the result holds for smaller values of $N(\gamma)$. We now analyze two cases.

Case 1. The sequence γ does not contain the segment 10. In this case we clearly have that $\gamma = 0^k 1^l$ for some $k, l \geqslant 0$. If $l \geqslant 1$, using the fact that $\mathcal{M}(\eta 1) = \mathcal{M}(\eta)\, 1$ for any $\eta \in \mathbf{Y}$ and the induction hypothesis, we are home. Thus we may assume that $\gamma = 0^k$ for some $k \geqslant 1$. Now note that

$$\mathcal{M}(\zeta 100^k) = \mathcal{M}(\zeta)\, 10^{k+1} \cup \mathcal{M}(\zeta 0)\, 10^k \cup \cdots \cup \mathcal{M}(\zeta 0^{k+1})\, 1 \tag{2.9}$$

where clearly the sets on the right-hand side are pairwise disjoint. Similarly, we have

$$\mathcal{M}(\zeta 10^k) = \mathcal{M}(\zeta)\, 10^k \cup \mathcal{M}(\zeta 0)\, 10^{k-1} \cup \cdots \cup \mathcal{M}(\zeta 0^k)\, 1 \tag{2.10}$$

with all the unions disjoint. We now observe that, by (2.10), the elements in all but the last set on the right-hand side of (2.9) are in natural one-to-one correspondence with the elements in $\mathcal{M}(\zeta 10^k)$. Moreover, since the elements in the last set on the right-hand side of (2.9) correspond to the elements in $\mathcal{M}(\zeta 0^{k+1}) = \mathcal{M}(\zeta 0\gamma)$ in an obvious way, we have that

$$M(\zeta 10\gamma) = |\mathcal{M}(\zeta 10^{k+1})| = |\mathcal{M}(\zeta 10^k)| + |\mathcal{M}(\zeta 0^{k+1})| = M(\zeta 1\gamma) + M(\zeta 0\gamma)$$

as required.

Case 2. The sequence γ contains the segment 10. In this case let us write $\gamma = \gamma_1 10\gamma_2$. Using the induction hypothesis, we have that

$$\begin{aligned} M(\zeta 10\gamma) &= M(\zeta 10\gamma_1 10\gamma_2) = M(\zeta 10\gamma_1 1\gamma_2) + M(\zeta 10\gamma_1 0\gamma_2) \\ &= M(\zeta 1\gamma_1 1\gamma_2) + M(\zeta 0\gamma_1 1\gamma_2) + M(\zeta 1\gamma_1 0\gamma_2) + M(\zeta 0\gamma_1 0\gamma_2) \\ &= M(\zeta 1\gamma_1 10\gamma_2) + M(\zeta 0\gamma_1 10\gamma_2) = M(\zeta 1\gamma) + M(\zeta 0\gamma) \end{aligned}$$

completing the induction step, and hence the proof. ∎

To close this section, we need to prove Lemma 2.4. To this end, we consider a function $R(\zeta)$ ($\zeta \in \mathbf{Y}$), implicit in ref. 4, which will turn out to give an alternative combinatorial description of the quantity $M(\zeta)$. It is using this description that we shall prove Lemma 2.4.

Let $W = (W_i)_1^n$ be a $\{\pm 1, 0\}$-sequence and $L = (L_i)_1^n$ a 0–1 sequence.

We say that (W, L) is a *labeled closed walk* of length n if (i) W is a *closed walk on* $\mathbb{Z}_+$ *starting at* 0, that is, if all initial partial sums $\sum_{1 \leqslant i \leqslant j} W_i$ $(0 \leqslant j \leqslant n)$ are nonnegative and $\sum_{1 \leqslant i \leqslant n} W_i = 0$, and (ii) L is such that, for all $1 \leqslant i \leqslant n$, if $W_i = 1$, then $L_i = 1$, if $W_i = -1$, then $L_i = 0$, and if $W_i = 0$, then $L_i \in \{0, 1\}$. For brevity, we refer to a closed walk on $\mathbb{Z}_+$ starting at 0 simply as a *closed walk*. Given $\zeta \in \mathbf{Y}$, let $\mathcal{R}(\zeta)$ be the set of all labeled closed walks (W, L) with $L = \zeta$, and put $R(\zeta) = |\mathcal{R}(\zeta)|$.

Proof of Lemma 2.4. We start by proving the following claim.

Claim. For all ζ, $\gamma \in \mathbf{Y}$, we have $R(\zeta 10\gamma) = R(\zeta 1\gamma) + R(\zeta 0\gamma)$.

Proof of the Claim. Let ζ, $\gamma \in \mathbf{Y}$ be fixed. Suppose $W = (W_i)_1^n$ is a closed walk for which $(W, \zeta 10\gamma)$ is a labeled closed walk. Assume $W = W^{(1)} w_1 w_2 W^{(2)}$, where $W^{(1)}$ and $W^{(2)}$ are $\{\pm 1, 0\}$-sequences of length $N(\zeta)$ and $N(\gamma)$, respectively, and w_1, $w_2 \in \{\pm 1, 0\}$. We put

$$\varphi(W, \zeta 10\gamma) = \begin{cases} (W^{(1)}0W^{(2)}, \zeta 1\gamma) & \text{if} \quad (w_1, w_2) = (0, 0) \\ (W^{(1)}0W^{(2)}, \zeta 0\gamma) & \text{if} \quad (w_1, w_2) = (1, -1) \\ (W^{(1)}1W^{(2)}, \zeta 1\gamma) & \text{if} \quad (w_1, w_2) = (1, 0) \\ (W^{(1)}(-1)\, W^{(2)}, \zeta 0\gamma) & \text{if} \quad (w_1, w_2) = (0, -1) \end{cases}$$

Then it is straightforward to check that φ defines a bijection between $\mathcal{R}(\zeta 10\gamma)$ and $\mathcal{R}(\zeta 1\gamma) \cup \mathcal{R}(\zeta 0\gamma)$, proving the claim. ■

Putting together the claim above and Lemma 2.3, we deduce that $R(\zeta) = M(\zeta)$ for all $\zeta \in \mathbf{Y}$, since $R(0^k 1^l) = M(0^k 1^l) = 1$ for all k, $l \geqslant 0$. We are now ready to start the proof of Lemma 2.4 proper. The calculations below, which are included for completeness, appear in the Appendix of ref. 4 in a slightly different form.

Let a and $b \geqslant 0$ be integers. For convenience, let us say that a 0–1 sequence L is of *type* (a, b) if L has a elements equal to 1 and b elements equal to 0. Let $r_{a,b}$ be the number of labeled closed walks (W, L) with L of type (a, b). Thus $r_{a,b} = \sum_\zeta R(\zeta) = \sum_\zeta M(\zeta)$, where the sum ranges over all ζ with $N(\zeta) = a + b$ and $K(\zeta) = a$. Moreover, if W_0 is a given closed walk, let $r_{W_0,a,b}$ be the number of labeled closed walks (W, L) with $W = W_0$ and L a sequence of type (a, b). Clearly $r_{a,b} = \sum_W r_{W,a,b}$, where the sum ranges over all closed walks W of length $a + b$.

The easiest way of handling the numbers $r_{a,b}$ and $r_{W,a,b}$ is by using generating functions. In the sequel, we shall consider bivariate formal power series with formal variables x and y. Let $n \geqslant 0$ be an integer. We put $\psi_n(x, y) = \sum_{a,b} r_{a,b} x^a y^b$, where the sum ranges over all pairs (a, b) with a, $b \geqslant 0$ and $a + b = n$. Moreover, for a closed walk W, we put $\psi_W(x, y) = \sum_{a,b \geqslant 0} r_{W,a,b} x^a y^b$. Then clearly $\psi_n(x, y) = \sum_W \psi_W(x, y)$, where the sum is over all closed walks W of length n.

Now, if a closed walk $W=(W_i)_1^n$ has $2q$ nonzero entries, it is immediate that we have $\psi_W(x, y)=(x+y)^{n-2q}x^q y^q$. Now note that the number of closed walks $W=(W_i)_1^n$ of length n with $2q$ nonzero entries is

$$(q+1)^{-1}\binom{2q}{q}\binom{n}{2q}$$

Indeed, to each such walk W, associate the walk $W'=(W_i')_1^{n+1}$ with $W_i'=W_i$ for $1\leqslant i\leqslant n$ and $W_{n+1}'=-1$. Then all proper partial initial sums of W' are nonnegative and $\sum_{1\leqslant i\leqslant n+1}W_i'=-1$. The number of such sequences W' is

$$(n+1)^{-1}\binom{2q+1}{q}\binom{n+1}{2q+1}$$

Choose where to have the ± 1 in W' randomly, and then Raney's lemma (cf. the proof of Lemma 2.6) tells us that a fraction of $1/(n+1)$ of such choices will do for W'. Thus the number of closed walks of length n and $2q$ nonzero entries is

$$\begin{aligned}\frac{1}{n+1}\binom{2q+1}{q}\binom{n+1}{2q+1}&=\frac{1}{2q+1}\binom{2q+1}{q}\binom{n}{2q}\\&=\frac{1}{q+1}\binom{2q}{q}\binom{n}{2q}\end{aligned}$$

as claimed. (Here and in the sequel the reader is referred to Chapter 5 of ref. 10 for identities involving binomial coefficients.) Therefore

$$\begin{aligned}\psi_n(x, y)&=\sum_q\frac{1}{q+1}\binom{2q}{q}\binom{n}{2q}(x+y)^{n-2q}x^q y^q\\&=\sum_{q,j}\frac{1}{q+1}\binom{2q}{q}\binom{n}{2q}\binom{n-2q}{j}x^{q+j}y^{n-(q+j)}\\&=\sum_{q,k}\frac{1}{q+1}\binom{2q}{q}\binom{n}{2q}\binom{n-2q}{k-q}x^k y^{n-k}\\&=\sum_{q,k}\frac{1}{q+1}\binom{n}{k}\binom{k}{q}\binom{n-k}{q}x^k y^{n-k}\\&=\sum_k\frac{1}{k+1}\binom{n}{k}x^k y^{n-k}\sum_q\binom{k+1}{q+1}\binom{n-k}{q}\\&=\sum_k\frac{1}{k+1}\binom{n}{k}\binom{n+1}{k}x^k y^{n-k}\end{aligned}$$

Therefore we have that

$$r_{k,n-k} = (k+1)^{-1} \binom{n}{k} \binom{n+1}{k}$$

and hence Lemma 2.4 follows. ■

We close this section with the following remark. Recall that $M(\zeta)$ appears in the definition of the probability measure $p = p_{\rho,\lambda}$ on $\mathbf{Y}$, and that we shall use p to construct invariant measures for our two-species asymmetric processes. We feel that the definition of $M(\zeta)$ makes it natural that this quantity should be involved in our construction. The alternative description of $M(\zeta)$ as a certain number of labeled walks on $\mathbb{Z}_+$, given in the proof of Lemma 2.4 above, allows us to perform some calculations, and in particular to prove Lemma 2.4.

3. INVARIANT MEASURES FOR THE PROCESS AS SEEN FROM A SECOND-CLASS PARTICLE

In the sequel, we shall always have $\rho \leqslant \lambda$. Given $0 < \rho \leqslant \lambda < 1$, we construct here a "translation-invariant" measure μ_2', in the sense this measure is invariant under translations that leave a second-class particle at the origin. The parameter ρ corresponds to the asymptotic density of the first-class particles, and λ corresponds to the asymptotic density of all the particles, with classes disregarded. In Proposition 1 we show that under μ_2' the distribution of first-class particles to the right of the origin and the distribution of empty sites to the left of it are product measures with densities ρ and $1-\lambda$, respectively. Another important and nice property of the measure μ_2' is that the distribution of the distance between two successive second-class particles is the same as the distribution of the hitting time of 1 for the random walk $\tilde{Z}_n$ introduced after Lemma 2.3. This observation and Proposition 1 give an alternative way of computing the decay of densities found by Derrida *et al.*[4] (see Remark 3.2 below).

In Theorem 1 we show that μ_2' is invariant for the process. We then construct other invariant measures for the process as seen from a second-class particle, randomly drawing a configuration according to μ_2' and identifying first- and second-class particles to the right of the origin and empty sites and second-class particles to the left of it. In particular, we get the shocks when $\lambda > \rho$: the invariant measure as seen from a single, isolated second-class particle. We may also obtain an invariant measure with only two second-class particles. If $\lambda = \rho$, the distance between these two particles is a nondegenerate random variable with an infinite first moment. In this case the corresponding random walk $\tilde{Z}_n$ is *symmetric* and the hitting time of 1 is finite with probability one but has an infinite mean.

Let $\{\zeta_i\}_{i\in\mathbb{Z}}\subset\mathbf{Y}$ be a doubly infinite i.i.d. sequence of finite configurations with distribution $\mathbb{P}(\zeta_i=\zeta)=p(\zeta)$, where $p(\zeta)$ is given in (2.1). A configuration (σ,ξ) with distribution μ_2' is obtained by displaying the ζ_i on the integers separated by second-class particles. More rigorously, for $i\geqslant 0$, let $N_i=N(\zeta_i)+1$ and $S_i=\sum_{j=0}^{i-1}N_j$. Let $I(x)=i$ if and only if $S_i\leqslant x<S_{i+1}$ $(i\in\mathbb{Z})$. Set $\sigma(0)=0$, $\xi(0)=1$, and for $x>0$ put

$$\sigma(x)=\begin{cases}\zeta_{I(x)}(x-S_{I(x)}) & \text{if} \quad S_{I(x)}<x<S_{I(x)+1}\\ 0 & \text{if} \quad x=S_{I(x)}\end{cases}$$

$$\xi(x)=\begin{cases}1 & \text{if} \quad x=S_{I(x)}\\ 0 & \text{otherwise}\end{cases}$$

Define $\sigma(x)$ and $\xi(x)$ for $x<0$ analogously. The resulting distribution of (σ,ξ) is the measure μ_2' that we seek.

Theorem 1. Let $0<\rho\leqslant\lambda<1$. The measure μ_2' is invariant for $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$, the process as seen from a second-class particle.

Before proving the theorem above, we construct other invariant measures using μ_2' and identification operators. Let $\xi=\{x_i\}_{i\in\mathbb{Z}}$ be the set of occupied sites of a configuration ξ of second-class particles with the origin occupied, where $x_0=0$ and $x_i<x_{i+1}$ for all integers i. Let Φ_i and Ψ_i $(i\in\mathbb{Z})$ be operators on configurations (σ,ξ) defined by setting, for all $x\in\mathbb{Z}$,

$$\Phi_i(\sigma(x),\xi(x))=\begin{cases}(\sigma(x)+\xi(x),0) & \text{if} \quad x\geqslant x_i\\ (\sigma(x),\xi(x)) & \text{otherwise}\end{cases}$$

$$\Psi_i(\sigma(x),\xi(x))=\begin{cases}(\sigma(x),0) & \text{if} \quad x\leqslant x_i\\ (\sigma(x),\xi(x)) & \text{otherwise}\end{cases}$$

In words, Φ_i identifies first- and second-class particles to the right of the ith second-class particle and Ψ_i identifies empty sites and second-class particles to the left of the ith second class particle. The next lemma, which is a straightforward generalization of an observation in Ferrari *et al.*,[9] says that the identification operators commute with the process as seen from a second-class particle. The reason is that, owing to the nearest-neighbor interaction rules, namely, the total asymmetry of the jumps and the exclusion interaction, the second-class particles to the right of a given second-class particle behave as though they were first-class particles. For the same reason, second-class particles to the left of a given second-class particle are just like empty sites.

Lemma 3.1. For any $i>0$ and $j<0$, the operators Φ_i and Ψ_j commute with the generator L'_2 of the process as seen from the second-class particle:

$$\Phi_i L'_2 = L'_2 \Phi_i, \qquad \Psi_j L'_2 = L'_2 \Psi_j$$

An immediate corollary of Lemma 3.1 is the following. Let $\Phi_\infty = \Psi_{-\infty} = I$, the identity operator.

Theorem 2. For any $0<\rho\leqslant\lambda<1$ and any $0<i\leqslant\infty$ and $-\infty\leqslant j<0$ the measures

$$\mu'_{2,i,j} = \Phi_i \Psi_j \mu'_2$$

are invariant for the process as seen from a second-class particle.

Remark 3.1. For all $0<\rho\leqslant\lambda<1$, the measure $\mu'_{2,1,-1}$ is the measure as seen from an isolated second-class particle. Moreover, the measure $\mu'_{2,2,-1}$ is a measure with only two second-class particles.

Proof of Theorem 1. By a standard construction of the process,[13,1] it is sufficient to verify the equality

$$\int L'_2 f(\sigma,\xi)\, d\mu'_2(\sigma,\xi) = 0 \tag{3.1}$$

for $f = 1_\Gamma\{(\sigma,\xi)_\Lambda\}$, where Λ is a finite subset of $\mathbb{Z}$ containing the origin, $(\sigma,\xi)_\Lambda$ is the projection of the configuration (σ,ξ) in Λ, and Γ is an arbitrary configuration of the form

$$2\gamma_{-k}2\gamma_{-k+1}2\cdots 2\gamma_{-1}2\gamma_1 2\cdots 2\gamma_k 2$$

with arbitrary k and arbitrary $\gamma_i \in \mathrm{Y}$ $(i=-k, -k+1,..., -1, 1,..., k)$. Put $l_- = -\sum_{i=-k}^{-1}[N(\gamma_i)+1]$ and $l_+ = \sum_{i=1}^{k}[N(\gamma_i)+1]$. Then, here, we have $\Lambda = \{x\in\mathbb{Z}: l_- \leqslant x \leqslant l_+\}$. Moreover $1.\{\cdot\}$ is the usual indicator function and

$$\begin{aligned} L'_2 f(\sigma,\xi) = \sum_{x\neq 0} &\{\sigma(x)[1-\sigma(x+1)][f(\sigma^{x,x+1},\xi^{x,x+1}) - f(\sigma,\xi)] \\ &+ \xi(x)[1-\sigma(x+1)][1-\xi(x+1)][f(\sigma,\xi^{x,x+1}) - f(\sigma,\xi)]\} \\ &+ \sigma(-1)[f(\tau_{-1}\sigma^{-1,0},\tau_{-1}\xi^{-1,0}) - f(\sigma,\xi)] \\ &+ [1-\sigma(1)][1-\xi(1)][f(\tau_1\sigma^{0,1},\tau_1\xi^{0,1}) - f(\sigma,\xi)] \end{aligned}$$

Let μ_2'', μ_2''' and μ_2'''' be the projections of μ_2' on Λ, $\{l_- - 1\} \cup \Lambda$, and $\Lambda \cup \{l_+ + 1\}$, respectively. Then verifying (3.1) amounts to verifying the equality of the following two expressions:

$$\mu_2'''(1\Gamma) + \sum_{x \in \Lambda_1} \mu_2''(\Gamma) + \mu_2''''(\Gamma 0) \tag{3.2}$$

and

$$\mu_2'''(20\Gamma_-) + \sum_{x \in \Lambda_2} \mu_2''(\Gamma_{x,x+1}) + \mu_2''''(\Gamma^- 12) \tag{3.3}$$

where

$$\begin{aligned}
\Lambda_1 &= \{x \in \Lambda\colon \Gamma(x) = 1, \Gamma(x+1) = 0 \text{ or } 2\} \\
&\quad \cup \{x \in \Lambda^-\colon \Gamma(x) = 2, \Gamma(x+1) = 0\} \\
\Lambda_2 &= \{x \in \Lambda\colon \Gamma(x) = 0 \text{ or } 2, \Gamma(x+1) = 1\} \\
&\quad \cup \{x \in \Lambda\colon \Gamma(x) = 0, \Gamma(x+1) = 2\} \\
\Lambda^- &= \Lambda \backslash \{l_+\} \\
\Gamma_- &= \gamma_{-k} 2 \gamma_{-k+1} 2 \cdots 2 \gamma_{-1} 2 \gamma_1 2 \cdots 2 \gamma_k 2 \\
\Gamma^- &= 2 \gamma_{-k} 2 \gamma_{-k+1} 2 \cdots 2 \gamma_{-1} 2 \gamma_1 2 \cdots 2 \gamma_k \\
\Gamma_{x,x+1} &= \{\Gamma(y),\ y \in \Lambda,\ y < x\}\ \Gamma(x+1)\ \Gamma(x) \{\Gamma(y),\ y \in \Lambda,\ y > x+1\}
\end{aligned}$$

We first show $\mu_2'''(1\Gamma) = \mu_2''''(\Gamma^- 12)$. Notice that

$$\mu_2'''(1\Gamma) = \mu_2'(\sigma(-1) = 1)\, \mu_2''(\Gamma) = \rho\lambda\mu_2''(\Gamma)$$

On the other hand,

$$\mu_2''''(\Gamma^- 12) = \prod_{i=-k}^{k-1} p(\gamma_i) \times p(\gamma_k 1) = \prod_{i=-k}^{k-1} p(\gamma_i) \times p(\gamma_k)\, \rho\lambda = \rho\lambda\mu_2''(\Gamma)$$

where p is the probability measure given by (2.1). Similarly, $\mu_2''''(\Gamma 0) = \mu_2'''(20\Gamma_-)$, so we only need to show that the two central sums in (3.2) and (3.3) are equal. The first thing to notice is that $\mu_2''(\Gamma)$ factors in the following way:

$$\mu_2''(\Gamma) = m(\Gamma) \times \lambda(1-\rho)(\lambda\rho)^{k(\Gamma)}\, [(1-\lambda)(1-\rho)]^{n(\Gamma) - k(\Gamma)} \tag{3.4}$$

where $m(\Gamma) = \prod_{i=-k}^{k} M(\gamma_i)$, $k(\Gamma) = \sum_{i=-k}^{k} K(\gamma_i)$, and $n(\Gamma) = \sum_{i=-k}^{k} N(\gamma_i)$. The measure $\mu_2''(\Gamma_{x,x+1})$ factors in a similar way when $x \in \Lambda^- \backslash \{\sum_{i=-k}^{-1} [N(\gamma_i) + 1]\}$. If, moreover, $x \in \Lambda_2$, then $k(\Gamma) = k(\Gamma_{x,x+1})$

and $n(\Gamma)=n(\Gamma_{x,x+1})$. So the last factor in the product (3.4) for $\mu_2''(\Gamma)$ equals the corresponding one in $\mu_2''(\Gamma_{x,x+1})$.

When $l_- \in \Lambda_2$, we have

$$\mu_2''(\Gamma_{x,x+1}) = m(\Gamma_{x,x+1}) \times \lambda(1-\rho)(\lambda\rho)^{k(\Gamma_{x,x+1})} \\ \times [(1-\lambda)(1-\rho)]^{n(\Gamma_{x,x+1})-k(\Gamma_{x,x+1})} \times \mu_2'(\sigma(-1)=1)$$

Since in this case $k(\Gamma_{x,x+1}) = k(\Gamma) - 1$, $n(\Gamma_{x,x+1}) = n(\Gamma) - 1$, and $\mu_2'(\sigma(-1)=1)=\lambda\rho$, the product of the last two terms in the above expression equals the last term in (3.4). A similar thing happens when $l_+ \in \Lambda_2$. Hence the factors dependent on λ and ρ in the terms of both central sums in (3.2) and (3.3) are the same and so it is sufficient to verify

$$\sum_{x \in \Lambda_1} m(\Gamma) = \sum_{x \in \Lambda_2} m(\Gamma_{x,x+1}) \tag{3.5}$$

This is proven in exactly the same way as (3.5) in ref. 4 (indeed, m here is the same object as w in that paper), by observing the following properties of m, which are inherited from M:

$$m(\Gamma'10\Gamma'') = m(\Gamma'1\Gamma'') + m(\Gamma'0\Gamma'')$$

$$m(\Gamma'12\Gamma'') = m(\Gamma'2\Gamma''), \qquad m(\Gamma'20\Gamma'') = m(\Gamma'2\Gamma'')$$

The conclusion is that both sides in (3.5) equal

$$\sum_{j: y_j = 1,0} m(y_1^{k_1} \cdots y_j^{k_j - 1} \cdots y_m^{k_m})$$

where $y_i^{k_i}$ $(i=1,..., m)$ are the maximal blocks of y_i $(y_i \in \{0, 1, 2\})$ constituting Γ, that is, $\Gamma = y_1^{k_1} \dots y_j^{k_j} \dots y_m^{k_m}$. ■

The next result was announced by Derrida *et al.*[4] and proved by Speer[17] through direct computations. Our proof is based on constructing the measure μ_2' by first displaying the first-class particles to the right of the origin according to a product measure and then specifying the positions of the second-class particles. The same is done to the left of the origin with the empty sites.

Proposition 1. Under μ_2', the distribution of first-class particles to the right of the origin is the product measure ν_ρ of parameter ρ. Similarly, the distribution of holes to the left of the origin is the product measure $\nu_{1-\lambda}$ with parameter $1-\lambda$.

Proof. Let a configuration $\eta \in \{0, 1\}^{\mathbf{Z}}$ be given. For $x \geqslant 1$, let $\eta|_x \in \mathbf{Y}$ be the finite configuration $\eta(1)\eta(2)\cdots\eta(x-1)$ of length $x-1$ determined by η. For all $x \geqslant 1$, put $M(\eta, x) = M(\eta|_x)$ and similarly put $K(\eta, x) = K(\eta|_x)$. For $x \geqslant 1$, let

$$p(x \mid \eta) = \begin{cases} 0 & \text{if} \quad \eta(x) = 1 \\ \lambda M(\eta, x)\, \lambda^{K(\eta, x)}(1-\lambda)^{x - K(\eta, x) - 1} & \text{if} \quad \eta(x) = 0 \end{cases} \tag{3.6}$$

Notice that $p(x \mid \eta)$ depends on η only through sites $1,..., x$. To prove our result it suffices to prove that

$$\sum_{x > 0} p(x \mid \eta) = 1 \tag{3.7}$$

ν_ρ almost surely, which is proven in Lemma 3.2 below. The reason is that we can interpret $p(x \mid \eta)$ as the probability that the leftmost second-class particle to the right of the origin is at site x given that the first-class particles are at the sites occupied by η. To see this, compute, for instance, the probability that the configuration $\zeta = 11010$ appears between the second-class particle at the origin and the next second-class particle (at site 6). According to our construction, first distribute three first-class particles and two holes at sites $\{1,..., 5\}$ with probability $\rho^3(1-\rho)^2$. Then put a hole at site 6 with probability $1-\rho$. Finally, the conditional probability of putting a second-class particle at site 6 given the configuration 110100... is

$$p(6 \mid 110100...) = \lambda M(11010)\, \lambda^3 (1-\lambda)^2$$

The resulting distribution is exactly the one given by (2.1). This argument can be applied to an arbitrary configuration, but the notation is too heavy, and hence we omit the details. ■

Lemma 3.2. Let $p(x \mid \eta)$ be defined as in (3.6). Then for all $\rho \in [0, \lambda]$

$$\sum_{x > 0} p(x \mid \eta) = 1 \tag{3.8}$$

ν_ρ almost surely. Furthermore, (3.8) holds for all configurations $\eta \in \{0, 1\}^{\mathbb{Z}}$ with a finite number of particles.

Proof. We first prove the identity (3.8) for configurations $\eta \in \{0, 1\}^{\mathbb{Z}}$ with a finite number of particles. Observe that if $\eta(x) = 0$ for all $x > 0$, then

$$\sum_{x > 0} p(x \mid \eta) = \lambda \sum_{x > 0} (1-\lambda)^{x-1} = 1$$

Assume that the identity holds for any configuration with n particles. Let η be a configuration with $n+1$ particles, whose rightmost particle is located at $z > 0$. Let η^z be the configuration η modified only at site z. Hence η^z has n particles. From Lemma 2.3, for $x \geqslant z+2$, we have

$$M(\eta, x) = M(\eta^z, x-1) + M(\eta, x-1) \tag{3.9}$$

Divide the sum in (3.8) in two parts:

$$\sum_{x>0} p(x \mid \eta) = \sum_{x=1}^{z+1} p(x \mid \eta) + \sum_{x=z+2}^{\infty} p(x \mid \eta) \tag{3.10}$$

Apply identity (3.9) to all terms of the second sum of the right-hand side of (3.10) to obtain

$$\begin{aligned}\sum_{x=z+2}^{\infty} p(x \mid \eta) = &\sum_{x=z+2}^{\infty} [1-\eta(x)]\, M(\eta^z, x-1)\, \lambda^{K(\eta,x)+1}(1-\lambda)^{x-1-K(\eta,x)} \\ &+ \sum_{x=z+2}^{\infty} [1-\eta(x)]\, M(\eta, x-1)\, \lambda^{K(\eta,x)+1}(1-\lambda)^{x-1-K(\eta,x)}\end{aligned}$$

Since for $x \geqslant z+2$,

$$K(\eta, x) = K(\eta, x-1) = K(\eta^z, x-1) + 1 \quad \text{and} \quad 1-\eta(x) = 1-\eta(x-1) = 1$$

we obtain that this sum is equal to

$$\begin{aligned}&\lambda \sum_{x=z+2}^{\infty} [1-\eta(x-1)]\, M(\eta^z, x-1)\, \lambda^{K(\eta^z, x-1)+1}(1-\lambda)^{(x-1)-1-K(\eta^z, x-1)} \\ &\quad + (1-\lambda) \sum_{x=z+2}^{\infty} [1-\eta(x-1)]\, M(\eta, x-1)\, \lambda^{K(\eta, x-1)+1} \\ &\quad \times (1-\lambda)^{(x-1)-1-K(\eta, x-1)}\end{aligned}$$

Hence

$$\sum_{x=z+2}^{\infty} p(x \mid \eta) = \lambda \sum_{x=z+1}^{\infty} p(x \mid \eta^z) + (1-\lambda) \sum_{x=z+1}^{\infty} p(x \mid \eta) \tag{3.11}$$

Observe that for $x \leqslant z$, $M(\eta, x) = M(\eta^z, x)$, $K(\eta, x) = K(\eta^z, x)$, while $1-\eta(x) = 1-\eta^z(x)$ for $x < z$ and $1-\eta(z) = 0$. Hence, multiplying by $(1-\lambda)+\lambda$ the first z terms of the first sum in the right-hand side of (3.10), we obtain

$$\sum_{x=1}^{z+1} p(x \mid \eta) = (1-\lambda) \sum_{x=1}^{z} p(x \mid \eta)$$
$$+ \lambda \sum_{x=1}^{z-1} [1-\eta^{z}(x)] \, M(\eta^{z}, x) \, \lambda^{K(\eta^{z},x)+1} (1-\lambda)^{x-1-K(\eta^{z},x)}$$
$$+ [1-\eta(z+1)] \, M(\eta, z+1)$$
$$\times \lambda^{K(\eta,z+1)+1} (1-\lambda)^{(z+1)-K(\eta,z)-1} \tag{3.12}$$

Now

$$M(\eta, z+1) = M(\eta^{z}, z)$$
$$K(\eta, z+1) = K(\eta^{z}, z) + 1$$
$$1-\eta(z+1) = 1-\eta^{z}(z) = 1$$

Hence the last line equals $\lambda p(z \mid \eta^{2})$ and the second plus the third line equals $\lambda \sum_{x=1}^{z} p(x \mid \eta^{z})$. So, putting together (3.11) and (3.12), we get

$$\sum_{x>0} p(x \mid \eta) = (1-\lambda) \sum_{x>0} p(x \mid \eta) + \lambda \sum_{x>0} p(x \mid \eta^{z}) \tag{3.13}$$

Since the second sum in the right-hand side of (3.13) is one by the inductive hypothesis, this completes the induction step. Thus the result holds for finite η. Since the sum of the first n terms in (3.8) depends only on $\eta(1),..., \eta(n)$, the validity of (3.8) for finite η implies that for any η and $n \geqslant 1$, we have $\sum_{x=1}^{n} p(x \mid \eta) \leqslant 1$, which in turns implies that for any η

$$\sum_{x=1}^{\infty} p(x \mid \eta) = c(\eta) \leqslant 1 \tag{3.14}$$

Assume that there exists a set $\mathbf{X}_0$ with positive ν_ρ probability such that if $\eta \in \mathbf{X}_0$, then $c(\eta) \leqslant c < 1$. This and (3.14) imply that

$$1 > \int d\nu_\rho(\eta) \sum_{x=1}^{\infty} p(x \mid \eta) = \mathbb{P}(N < \infty) = 1 \qquad \text{for} \quad \rho \leqslant \lambda$$

where N is the random variable whose distribution is given by (2.1) and (2.3). The contradiction above proves that (3.8) holds ν_ρ almost surely for any $\rho \in [0, \lambda]$. ■

Remark 3.2. Using the generating function of N given in (2.3), one can estimate precisely the rate of convergence of the densities of the particles computed by Derrida *et al.*[4] Let $(\tilde{Z}_n)_1^\infty$ be the random walk on $\mathbb{Z}$ defined after Lemma 2.3. We say that there is a *record* at time $n \geqslant 1$ if $\tilde{Z}_n > \tilde{Z}_j$ for all $0 \leqslant j < n$. Hence the probability that under μ'_2 a second-class particle is present at site $n \geqslant 1$ is the probability that the random walk $(\tilde{Z}_n)_1^\infty$ establishes a record at time n. Put $u_0 = 1$, and for $n \geqslant 1$ let $u_n = (\lambda - \rho) + \rho(1 - \lambda)\, \mathbb{P}(T \geqslant n)$, where T is the time of the first record or, in other words, the hitting time of 1. Now (2.3), the relation between N and T, and the renewal equation[5] imply that u_n is precisely the probability that a record is established at time n. From the rate of convergence of the distribution of T (see remarks to the proofs of Lemmas 2.2 and 2.5) it is clear that u_n goes exponentially fast to $\lambda - \rho$ when $\lambda > \rho$ and like $n^{-1/2}$ when $\lambda = \rho$. Since the density of first-class particles to the right of the origin is a product measure with constant density ρ, this gives also the asymptotic density of holes to the right of the origin. Analogous arguments work to the left of the origin by observing that the density of holes is $1 - \lambda$.

4. THE INVARIANT MEASURE FOR THE TRANSLATION-INVARIANT PROCESS

In this section we assume $0 < \rho < \lambda < 1$. Let μ_2 be the unique translation-invariant measure satisfying $\mu_2(\cdot \mid \xi(0) = 1) = \mu'_2(\cdot)$. As mentioned in the introduction, μ_2 must be invariant for the two-species process. We show next that the measure μ_2 has good marginals.

Theorem 3. The σ marginal of μ_2 is ν_ρ, while the $\sigma + \xi$ marginal of μ_2 is ν_λ.

Proof. To construct the measure μ'_2 we started by assigning the positions of the second-class particles and then we gave the distribution of the first-class particles, given the position of the second-class particles. The positions of the second-class particles form a (discrete-time) renewal process with finite mean interarrival time, with the first renewal at time 0. When $\rho < \lambda$ the average distance between two renewals is $(\lambda - \rho)^{-1} < \infty$. Hence we can use the key renewal theorem to construct μ_2 in the following way:

$$\mu_2 = \lim_{x \to -\infty} \mu'_2 \tau_x = \lim_{x \to +\infty} \mu'_2 \tau_x \tag{4.1}$$

where τ_x is the translation by the x operator. To show the theorem, take a cylinder function $f(\sigma, \xi)$ depending only on σ. Take a negative z such that the support of f is contained in (z, ∞). By Proposition 1,

$$\mu'_2 \tau_z f = \nu_\rho f$$

This and (4.1) imply that $\mu_2 f = \nu_\rho f$. To show that the $\sigma + \xi$ marginal is ν_λ, apply the same reasoning for a positive z and show that $\mu_2 f = \nu_\lambda$ if f depends only on $\sigma + \xi$. ■

Our next result exploits the embedded reneval process in both μ_2 and μ_2'.

Theorem 4. It is possible to construct a coupling $\tilde{\mu}_2$ with marginals μ_2 and μ_2' such that if $(\sigma, \xi, \sigma', \xi')$ has distribution $\tilde{\mu}_2$ and

$$H(\sigma, \xi, \sigma', \xi') = \sum_x |\sigma(x) - \sigma'(x)| + |\xi(x) - \xi'(x)|$$

is the number of sites where (σ, ξ) is different from (σ', ξ'), then, under $\tilde{\mu}_2$, the random variable H has a finite exponential moment. In other words, there exists $\theta > 0$ such that

$$\int d\tilde{\mu}_2 \, e^{\theta H} < \infty \tag{4.2}$$

Proof. Let (σ, ξ) be a realization of the translation-invariant point process related to the point process with distribution μ_2' and T_i the stationary process related to ξ. Thus T_i denotes the position of the ith ξ particle, where $T_0 \leqslant 0$ is the position of the rightmost ξ particle to the left of the origin. Let $(\bar{\sigma}', \bar{\xi}')$ be a realization of the process with distribution μ_2' and let S_i be the renewal process associated to $\bar{\xi}'$, with $S_0 = 0$ and S_i denoting the position of the ith $\bar{\xi}'$ particle. The random variables $T_i - T_{i-1}$ are independent and have the same distribution as $S_i - S_{i-1}$ for $i \neq 1$, while (T_0, T_1) has the limiting distribution

$$\mathbb{P}(T_1 > u, -T_0 > v) = \lim_{t \to \infty} \mathbb{P}(S_{I(t)+1} - t > u, t - S_{I(t)} > v)$$

where $I(t) = \max\{i: S_i < t\}$. Similarly, for all cylinder f, we have $\mu_2 f = \lim_{x \to \infty} \mu_2' \tau_x f$. Now we construct a coupling $(\sigma, \xi, \sigma', \xi')$ with the property that the two first marginals have distribution μ_2 and the two last marginals have distribution μ_2'. If $T_0 = 0$, put $\xi'(x) \equiv \xi(x)$ and $\sigma'(x) \equiv \sigma(x)$. If $T_0 \neq 0$, let $J^+ = \min\{i > 0: S_i \in \xi\}$ and $J^- = \max\{i \leqslant 0: S_i \in \xi\}$, and let

$$(\sigma'(x), \xi'(x)) = \begin{cases} (\sigma(x), \xi(x)) & \text{if } \; x \geqslant J^+ \; \text{ or } \; x \leqslant J^- \\ (\bar{\sigma}'(x), \bar{\xi}'(x)) & \text{if } \; J^- < x < J^+ \end{cases}$$

It is clear that the resulting distribution of $(\sigma, \xi, \sigma', \xi')$ has marginals μ_2 and μ_2'. To show (4.2), notice that under $\tilde{\mu}_2$, we have $H \leqslant J^+ - J^-$. Since T_0 and T_0' have a finite exponential moment, it follows from ref. 14, pp. 30–31, that both J^+ and $|J^-|$ have a finite exponential moment. ■

5. SHOCKS IN THE SIMPLE EXCLUSION PROCESS

If we take the process as seen from a second-class particle and identify particles of both classes to the right of the origin and second-class particles with holes to the left of it, we get η'_t, the simple exclusion process as seen from an isolated second-class particle. Rigorously, Lemma 3.1 says that the process $\eta'_t := \tau_{X_t}\eta_t = \Phi_1 \Psi_{-1} \tau_{X_t}(\sigma_t + \xi_t)$ is the SEP as seen from a second-class particle. We consider the shock measure constructed in the remark after Theorem 2 of Section 3. Let $\mu' = \Phi_1 \Psi_{-1} \mu'_2$. For $0 < \rho \leqslant \lambda < 1$, it follows from Theorem 2 that μ' is invariant for the process η'_t. Notice that X_t can be seen as either a tagged second-class particle for the (σ_t, ξ_t) process or as an isolated second-class particle for the η_t process. Our next result implies in particular that, for $0 < \rho < \lambda < 1$, the measure μ' is equivalent to $\nu_{\rho,\lambda}$, the product measure with densities ρ and λ to the left and right of the origin, respectively.

Theorem 5. If $0 < \rho < \lambda < 1$, it is possible to construct jointly the invariant measure μ and the product measure $\nu_{\rho,\lambda}$ in such a way that the number of sites where the configurations differ has a finite exponential moment.

Proof. First we construct a configuration with distribution $\nu_{\rho,\lambda}$ using two independent configurations with distribution μ_2. Let (σ^+, ξ^+) and (σ^-, ξ^-) be two independent realizations of μ_2 and (σ', ξ') a realization of μ'_2 independent of the other two. Define $\eta \in \{0, 1\}$ by letting

$$\eta(x) = \begin{cases} \sigma^+(x) + \xi^+(x) & \text{if } x \geqslant 0 \\ \sigma^-(x) & \text{if } x < 0 \end{cases}$$

for all $x \in \mathbb{Z}$. Then, by the marginal properties of μ_2 given by Theorem 3, it is easy to see that η constructed above has distribution $\nu_{\rho,\lambda}$. Here it is important that we take independent realizations of μ_2 to the right and left of the origin. Now couple (σ^+, ξ^+) with (σ', ξ') as in the proof of Theorem 4, letting J_+ be the leftmost positive site where (σ^+, ξ^+) is different from (σ', ξ'). Similarly, couple to the left of the origin letting J^- be the rightmost negative site where (σ^-, ξ^-) differs from (σ', ξ'). By the same argument as before, J^+ and J^- have a finite exponential moment. Hence, setting $\eta' = \Phi_1 \Psi_{-1}(\sigma', \xi')$, we get

$$\sum_x |\eta'(x) - \eta(x)| \leqslant J^+ - J^-$$

The result now follows again from Lindvall.[14] ■

An immediate consequence of the result above is the following.

Corollary. The measures μ' and $\nu_{\rho,\lambda}$ are equivalent, i.e., one is absolutely continuous with respect to the other.

Proof. Since under both measures all nonempty cylinder sets have positive probability, and nonempty sets of measure zero depend on infinitely many coordinates, the corollary follows from Theorem 4. ■

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REFERENCES

1. E. D. Andjel, Invariant measures for the zero range process, *Ann. Prob.* **10**:525–547 (1982).
2. L. Breiman, *Probability* (Addison-Wesley, Reading, Massachusetts, 1968).
3. A. De Masi, C. Kipnis, E. Presutti, and E. Saada, Microscopic structure at the shock in the asymmetric simple exclusion, *Stochastics* **27**:151–165 (1990).
4. B. Derrida, S. Janowsky, J. L. Lebowitz, and E. Speer, Exact solution of the totally asymmetric simple exclusion process: Shock profiles, *J. Stat. Phys.* **73**(5/6):813–842.
5. W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1968).
6. P. A. Ferrari, The simple exclusion process as seen from a tagged particle, *Ann. Prob.* **14**:1277–1290 (1986).
7. P. A. Ferrari, Shock fluctuations in asymmetric simple exclusion, *Prob. Theory Related Fields* **91**:81–101 (1992).
8. P. A. Ferrari and L. R. G. Fontes, Shock fluctuations in the asymmetric simple exclusion process, *Prob. Theory Related Fields*, to appear.
9. P. A. Ferrari, C. Kipnis, and E. Saada, Microscopic structure of travelling waves for the asymmetric simple exclusion process, *Ann. Prob.* **19**:226–244 (1991).
10. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics* (Addison-Wesley, Reading, Massachusetts, 1989).
11. W. Hoeffding, Probability inequalities for sums of bounded random variables, *J. Am. Stat. Assoc.* **58**:13–30 (1963).
12. T. M. Liggett, Coupling the simple exclusion process, *Ann. Prob.* **4**:339–356 (1976).

13. T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, Berlin, 1985).
14. T. Lindvall, *Lectures on the Coupling Method* (Wiley, 1992).
15. C. J. H. McDiarmid, On the method of bounded differences, in *Surveys in Combinatorics 1989*, J. Siemons, ed. (Cambridge University Press, Cambridge, 1989), pp. 148–188.
16. G. N. Raney, Functional composition patterns and power series reversion, *Trans. AMS* **94**:441–451 (1960).
17. E. R. Speer, The two species asymmetric simple exclusion process, in *Micro, Meso, and Macroscopic Approaches in Physics*, M. Fannes, C. Maes, and A. Verbeure, eds. (Plenum, New York, 1994), pp. 91–102.
18. F. Spitzer, *Principles of Random Walk*, 2nd ed. (Springer-Verlag, Berlin, 1976).